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SUMMARY

The general problem of dynamic population interactions between communities is mathematically formulated according to certain concrete and fundamental conservation principles and plausible and realistic potential and constitutive laws. It is shown that the formulation leads to a system of coupled, non-linear, ordinary first order differential equations which can be numerically integrated by classical techniques for the prescribed initial conditions. A plausible hypothetical problem about two initially underdeveloped communities is investigated under the influence of the external surroundings to the system; and the numerical solutions to the problem, representing the living conditions of the communities for all time, are presented. It is clearly established that the quantitative levels for the living conditions of the residents, the population densities and the migration phenomena between the totality of the communities in the system are all significant factors which control the population growth in any single community. Hence, it is concluded that the future human habitation problems of a community of interest cannot be predicted correctly without simultaneously taking into consideration the same problems in all the communities which are interacting in the complete urban system.

INTRODUCTION

It is univocally accepted that the totality of our present environmental problems are directly related to the populations and the associated living conditions of the residents of our urban communities. It is also realized that one of the fundamental reasons which led to the creation of these problems was the constant neglect of the essential quantitative mathematical modeling for predicting the human habitation phenomenon in the planning and the development of residential communities for the future.

In the last ten years, after the introduction of high speed digital computers for the analysis of vast amounts of available data, numerous studies were made for the creation of mathematical models for urban planning and development. An extensive survey of these investigations, which cover a wide spectrum of the pertinent effects that control the development of metropolitan communities, was edited by Hemmens [1].[1]

At present, the mathematical treatments of the problems associated with urban populations are generally separated into two basic categories.

The more popular and relatively more sophisticated "statistical approach" is mainly concerned with predicting the future trends of the behavior of populations based on the analysis of the pertinent, observable data which are continuously compiled at the particular locations of the communities. The extensive data for populations and population growth rates of the continents, the nations, the national geographical regions, the large metropolitan areas and the small urban communities, together with the discussions about the methods of analyses of the data, are documented in various references, e.g., [2], [3], [4], [5].

[1]Numbers in brackets designate similarly numbered references at the end of the paper.

The mathematical field of statistical analysis has attained such advanced levels of sophistication today that any type of extrapolation from a given set of population data can be accomplished to any possible degree of accuracy. However, since there exists a vast number of variables, depending on the human attitudes, environmental factors, etc., which affect the human habitation phenomenon, any prediction about the outcome of the population of a community, based solely on the previously compiled population data, may be grossly erroneous in representing the actual outcome of the population variations of a community. Without the existence of a plausible mathematical theory which can incorporate these pertinent effects, the results of the statistical extrapolations can only be valid if these additional variables and parameters remain constant during the compilation of the data and also during the period of interest for which the predictions are required. However, if these values change, which is indeed the more realistic case in our complex society, then the previously obtained statistical data cannot be employed for future predictions. Indeed, it becomes necessary to compile sufficient amounts of data under the newly established conditions before attempting to make any predictions for the outcomes. Consequently, the statistical analysis by itself becomes a rather inefficient, time consuming and expensive method of predicting the populations for the planning and the development of urban communities.

The less popular but historically older "analytical methods" are primarily based on the calculation of population growth of communities according to the classical mathematical equations of Biophysics. This approach which is considerably more popular among ecologists and biologists (Kormondy [6,7], Sauvy [8], Hazen [9], Bailey [10]) was first introduced by Verhulst [11] and later independently by Volterra [12,13] and Lotka

[14,15] for the analysis of the local population growth problems in lower evolutionary level biological species. The mathematical systems associated with these theories were extended by numerous authors, e.g., Nicholson [16], Chapman [17], Slobodkin [18], to incorporate the effects of the environment and also to a limited extent the possible competition between different species for existence.

The fundamental mathematical model for the distribution of populations in space was first introduced by Skellam [19] based on the random dispersal hypothesis for the migration of the elements of the populations. The significant extensions of this theory were later investigated by numerous authors, Landahl [20,21], Barakat [22] and Thompson and Weiss [23] to investigate the solutions of the associated mathematical systems which were considerably similar to the diffusion equations of classical physics. A recent study by Yuill [24] fundamentally applies the basic concepts of the aforementioned studies to the analysis of spatial distributions of population over the residential areas of large metropolitan communities.

The development of more sophisticated mathematical theories based on the interpretation of the biological phenomena along the lines of statistical mechanics was accomplished by Kerner [25, 26, 27, 28, 29]. These papers consider in detail the similarities between the theories of physics and biology, particularly in conjunction with the continuity equation for population density and the gradient of population effects for the migration phenomena of the biological species.

Considering the rather plausible proposition of Rashevsky [30], [31], [32], that there must exist a unifying mathematical theory in Biophysics which could represent the behavior of all biological species regardless of the evolution level, it is more conceivable that the extension of the mathematical theories of biophysics would be more successful than a solely statistical approach in analyzing the human population and habitation (Ekistics) problems in the communities of our modern society.

The present status of both statistical and analytical mathematical theories associated with the modeling of population growth phenomena in urban communities is considerably inferior to the status of the mathematical theories related to classical physical sciences in view of three fundamental and significant shortcomings which warrant special consideration.

Firstly, the predictions for the outcome of human populations in a community based on statistical considerations depend extensively on the amount of compiled data about the particular community. At present it is practically impossible to employ the results of the statistical analysis of the population of a metropolitan area to predict the outcome in a new, developing, small urban community; and conversely, it is impossible to employ the statistical results about a small community to predict the outcome of the population growth in a highly populated city. Consequently, the statistical data that is compiled about the human habitation phenomenon in a community is not universally applicable to all the communities regardless of the fact that the human behavior patterns of our present society in general have rather well observed and well classified overall characteristics.

The second fundamental shortcoming of the presently available mathematical formulations for population analyses, particularly for the analytical methods, is the necessity to consider a single community without taking into consideration the interactions of the communities through the significantly important effect of migration of the residents between the communities. Considering the undeniable fact that the migration of residents from one community to another decreases the population of the former and increases the population of the latter, any mathematical modeling which neglects this significant and fundamental migratory interaction effect cannot be successful in realistically predicting the population trends of urban communities.

It is clearly evident that the human habitation phenomenon does not solely depend on the population, but probably more significantly on the average living condition of the human individual in an urban community. Indeed, it is well established that the totality of our social problems in largely populated metropolitan cities can be attributed qualitatively to the existence of an inverse proportionality type relation between the increase of population density and the decrease of level of the living standard of the residents. Furthermore, the totality of the factors, which determine the living conditions of an individual as an overall variable, significantly affect the migratory phenomena between different communities. Consequently, a mathematical theory about the quantitative analysis of the populations must necessarily include the quantitative representation of the living conditions of the individuals of the population as a variable; and, indeed, it must simultaneously analyze the changes of this variable together with the population of the communities.

The significance of this effect was first discussed qualitatively by Lotka [14]; and in a later study by Beckmann [33], the quantitative mathematical formulations of the influences of per capita income and the desirability of location was investigated in conjunction with the population dispersion problems. Thompson and Weiss [23] introduced a modification in the form of a functional term in the mathematical theories of Skellam [19] and Landahl [20, 21] tried to discuss the influence of this term in the mathematical solutions. However, at present, there exists no theory in mathematical ekistics which can quantitatively incorporate the average standard of living of the residents of the communities in the analysis. This third fundamental shortcoming is of significant importance since it primarily controls the behavior patterns of the human individual in our materialistic, advanced modern society.

An alternate approach to the theoretical investigation of the human habitation phenomena in large urban communities is based on the adaptations of the already established dynamic models in related fields. These applications which fundamentally employ the well known principles of feedback-loop models can be formulated for direct computer coding for the analysis of complex urban problems. The fundamental study by Forrester [34] discusses in detail the important factors and variables which need to be considered to have a clear understanding of the problems associated with a large urban community. Although this study may be considered as a significant contribution in view of its success in the extensive application of the digital computer to the analysis of urban problems, it has a fundamental shortcoming since it clearly neglects the effects of the changes occuring in other communities which interact with the community under consideration through the migratory effects of of the residents. Indeed, the fundamental assumption of the theory, "the urban area in its limitless environment" cannot be accepted as realistic in view of the clearly evident importance of the mobility effect of the people resulting from the continuously changing living conditions in all the communities of our modern society.

The three significant shortcomings of the presently available mathematical theories for the analysis of the human habitation problems apply to the same degree of importance to the mathematical modeling of the behavior of the biological species of lower evolution level. In a recent study by Eraslan [35], a general mathematical theory in Biophysics was presented for the dynamic behavior of living species which quantitatively incorporates the spatial distributions, migratory motions and the "degree of individual advancement" of the biological species in the analysis of populations.

The purpose of this paper is to present a simple, preliminary mathematical theory which can be extended to any desired degree of sophisticated modeling, to establish the governing mathematical systems based on fundamental conservation principles and potential and constitutive laws for the analysis of the dynamic ekistics problems between urban communities by taking into consideration the significantly important factors of interaction, migration and the changes in the living conditions of the residents of the communities.

DEFINITION OF PERTINENT QUANTITIES AND MATHEMATICAL VARIABLES OF THE SYSTEM

As the case for any physical phenomenon, the mathematical formulation of human habitation also requires the construction of a transformation model by establishing concrete definitions of the mathematical quantities as representations of the physically observable characteristic properties of the general problem.

DEFINITION 1. "COMMUNITY C_n" is defined as a region, with either geographically defined or abstractly specified closed boundaries, where sufficiently countable number of humans reside and exist under certain observable standard of living conditions.

The definition is intentionally stated as general as possible such that the concept can be employed to represent any habitat regardless of size, i.e., small towns, counties, large cities or even countries are all equivalent under the definition of the community. For the general formulation of population ekistics problem, it is required that there exists $N \geq 1$ number of distinct communities $C_1, C_2, \ldots, C_N$ with observable living conditions for analysis.

DEFINITION 2. "AREA A_n" of a community C_n is defined as the net, measurable, geographical or conceptual area which is inhabited and is interior to the defined closed boundary of the community.

POSTULATE 1. The area A_n of a community C_n, if defined appropriately, remains constant for the analysis of human habitation phenomenon.

The statement of the postulate may be considered as a non-realistic approximation since the inhabited area of a community may change in size with the changes in the growth of the community. However, noting that in the definition of a community, the closed boundary is quite arbitrary, it is always possible to describe a sufficiently large area which will remain constant during the analysis of the development of the community.

DEFINITION 3. "EXTERNAL SURROUNDINGS C_e" is defined as an inhabited region satisfying the following conditions:

a. Exterior to the boundaries of any community C_n,

b. Grossly larger in size than any community C_n,

c. Interaction with any community C_n does not influence any noticable observable variations in the behavior characteristics of the inhabitants of this region.

The part c. of the definition is of significant importance since it establishes the assumption that, due to its grossly larger properties in comparison to the communities, any change occurring in this region depends only on time, and it is completely independent of the behavior characteristics of the communities.

The inclusion of the external surroundings C_e in the analysis of the human habitation problems is not necessarily essential, but rather it is of convenience for certain problems when the behavior of small communities is under consideration. However, it is important to emphasize that a region cannot be defined as the external surroundings unless all the conditions of the Definition 2 are satisfied.

<u>DEFINITION 4</u>. "POPULATION GROUP $S^{(m)}$" is defined as a part of the overall population of inhabitants, who live in the communities $C_1, C_2, \ldots, C_N$, with distinctly distinguishable behavior characteristics in comparison to other groups for the analysis of population ekistics problem in the overall system.

The general definition of the population group $S_n^{(m)}$ can be employed to represent each individual group based on a special, particular, characteristic, e.g., sex, age, race, religion, profession, associated with the group. It is essential to realize that Definition 4 implies that the distinct separation of the population into groups depends on the requirements for the analysis of the problem. Consequently, for simple, preliminary mathematical models, the population ekistics problems can be formulated based on a single population group for each community. If the results of such analyses are not acceptable as accurate representations of the phenomenon, then, more sophisticated and detailed models with numerous population groups $m = 1, 2, \ldots\ldots, M$, can be employed.

<u>DEFINITION 5</u>. "AVERAGE POPULATION GROUP DENSITY $\bar{\rho}_n^{(m)}$" is defined as the average number of inhabitants belonging to the population group $S_n^{(m)}$ and residing on per unit area of the overall area A_n of the community C_n.

It is important to realize that the principle quantity, $\bar{\rho}_n^{(m)}$, of the formulation is defined strictly as population density and not as the total number of inhabitants of the community since the living conditions of a community depends more significantly on the former than the latter.

<u>POSTULATE 2</u>. The average population group densities $\bar{\rho}_n^{(m)}$ are appropriate and sufficiently representative principal quantities for the description of the variations with time of the population characteristics and the effects in relation to overall human habitation phenomenon in community C_n.

The postulate represents the simplifying assumption which eliminates the necessity to consider the spatial variations of the population group densities over the areas of the communities. The assumption may also seem rather unrealistic for the analysis of the real population problems associated with a large metropolitan city with significant differences in population densities in the urban and suburban areas which are included in the geographically defined boundaries of the community. However, considering the particular advantage of the abstract definition of the boundary of the community according to Definition 1, a large metropolitan community can be subdivided into sufficiently many small communities C_n with approximately uniform, average population group densities $\bar{\rho}_n^{(m)}$. Hence, the proposed formulation can be employed for the analysis of the realistic population ekistic problems in all communities regardless of size by the appropriate modification of subdivision of the large communities.

As the consequence of the Postulate 2, it is immediately evident that the subsequent development of the theory will consider time t as the only independent variable in the analysis; hence, all the principal characteristic quantities in the mathematical formulation will be treated as dependent variable unknowns in terms of the only independent variable time t for the general problem.

DEFINITION 6. "GROUP POPULATION $P_n^{(m)}$" is defined as the number of inhabitants belonging to population group $S^{(m)}$ and residing in community C_n.

As consequences of definitions 2 and 5 for the area A_n and the average population group density $\bar{\rho}_n^{(m)}$, respectively, the group population $P_n^{(m)}$ becomes

$$P_n^{(m)}(t) = A_n \bar{\rho}_n^{(m)}(t) \tag{1}$$

DEFINITION 7. "POPULATION P_n" is defined as the net number of inhabitants of all population groups $S^{(m)}$ residing in community C_n, i.e.,

$$P_n(t) = \sum_{m=1}^{M} P_n^{(m)}(t) = \sum_{m=1}^{M} A_n \bar{\rho}_n^{(m)}(t) = A_n \sum_{m=1}^{M} \bar{\rho}_n^{(m)}(t) \tag{2}$$

The definitions 5 through 7 establish the essential representations of the characteristic quantities associated with human populations of the communities for the analysis of the general problem.

DEFINITION 8. "AVERAGE DEGREE OF INDIVIDUAL ADVANCEMENT $\bar{D}_n^{(m)}$" of a human being, belonging to the population group $S^{(m)}$ and residing in the community C_n, is defined as a conceptual characteristic quantity which completely defines the relative average living conditions and the well being of the individual.

It is of significant importance to realize that according to definition 8, degree of individual advancement is strictly a characteristic property of the individual rather than being associated with the community as the totality of its inhabitants. Consequently, regardless of conceptual quality of the definition, it is sufficient to decide on the various observable characteristics of the living conditions of a human being in society to represent the defined quantity.

Considering our modern society, one of the fundamental characteristics which govern the living conditions of a human being is the yearly income. Hence, for simple formulations of the population ekistic problems, it may suffice to employ this characteristic quantity to represent the living standard of the individual inhabitant. However, it is indeed evident that there exist numerous other characteristics, e.g., educational level, health, social standing, residential living conditions, etc., which need to be considered for the complete definition of the quantity. Consequently, the average degree of individual advancement $\overline{D}_n^{(m)}$ can be considered as a vector quantity with J number of components which represent all the pertinent characteristics of the living conditions of the individual inhabitant in a community in general.

<u>DEFINITION 9</u>. "COMPONENT OF AVERAGE DEGREE OF INDIVIDUAL ADVANCEMENT $\overline{D}_{n,j}^{(m)}$" is defined as a selected, pertinent, observable, characteristic quantity which represents the living conditions of a human being, on the average, belonging to the population group $S^{(m)}$ and residing in the community C_n.

<u>DEFINITION 10</u>. "MINIMUM BOUND $D_{jmin}^{(m)}$" of the component of average degree of individual advancement of a human being is defined as the level at which the necessity for the consideration of his existence is terminated in any community.

<u>DEFINITION 11</u>. "MAXIMUM BOUND $D_{jmax}^{(m)}$" of the component of average degree of individual advancement of a human being is defined as the relative optimum

level that a human being can attain in any community.

The definitions 8-11 mathematically establish the concrete representation of the average living condition of an inhabitant of a community by the associated vector quantity average degree of individual advancement $\bar{D}_n^{(m)}$ based on its components $\bar{D}_{n,j}^{(m)}$. The choice of the particular, pertinent, characteristics of the living conditions of the individual, to be represented by these components, strictly depends on the sociological and economical factors of the society. However, provided that these factors are specified, the definitions 10 and 11, for the minimum and the maximum bounds, are sufficient to establish the ranges of the quantities; hence, appropriate scales can be constructed for the unknown dependent variables, component of average degree of individual advancement $D_{n,j}^{(m)}$, according to quantitative mathematical definitions in the formulation.

DEFINITION 12. "COMPONENT OF ADVANCEMENT $G_{n,j}^{(m)}$" of the population group $S^{(m)}$ in the community C_n is defined as the net component of advancement of the totality of the individuals of the group which is associated with the particular component of average degree of individual advancement $\bar{D}_{n,j}^{(m)}$, i.e.,

$$G_{n,j}^{(m)}(t) = A_n \bar{\rho}_n^{(m)}(t) \bar{D}_{n,j}^{(m)}(t) \tag{3}$$

The principal quantity component of advancement $G_{n,j}^{(m)}$, in a sense, represents the overall level of the particular characteristic of living conditions for the population group $S^{(m)}$ in the community C_n. Consequently, for the particular population group $S^{(m)}$ to exist under the $G_{n,j}^{(m)}$ component of advancement conditions, it is essential that the community C_n indeed possesses the sufficient means to support the inhabitants at their level of existence. The significant importance of this argument is that is proposes to create a correspondence between the population, the individual living conditions and the limited support of the community to establish the fundamental relations for the general formulation of the behavior of the human habitation phenomenon.

DEFINITION 13. "AVERAGE RATE OF GENERATION OF POPULATION GROUP DENSITY $\bar{\dot{\rho}}_n^{(m)}$" is defined as the average rate of increase of the average population group density $\bar{\rho}_n^{(m)}$ solely due to the living conditions in the particular community C_n without the consideration of the influences of other communities.

It is significantly important to realize that definition 13 emphatically states that the generation rate is strictly a property of the community C_n; hence, the effects of the migratory motion between the communities must necessarily be excluded in the evaluation of the quantity.

The defined quantity represents the net increase of population density, i.e., the net difference between the birth rate and the death rate, only if the formulation assumes one single population group for simplified analyses. However, for sophisticated formulations with various population groups, e.g., age groups, each $\bar{\dot{\rho}}_n^{(m)}$ term must be formulated according to the increase of the population density of the particular group during an appropriately specified time period.

DEFINITION 14. "RATE OF GENERATION OF GROUP POPULATION $\dot{P}_n^{(m)}$" is defined as the net rate of increase of the group population $P_n^{(m)}$ in the community C_n, i.e.,

$$\dot{P}_n^{(m)}(t) = A_n \bar{\dot{\rho}}_n^{(m)}(t) \quad ; \quad \bar{\dot{\rho}}_n^{(m)}(t) = \dot{P}_n^{(m)}(t)/A_n \tag{4}$$

DEFINITION 15. "RATE OF GENERATION OF POPULATION $\dot{P}_n$" is defined as the net rate of increase of the population P_n of community C_n, i.e.,

$$\dot{P}_n(t) = \sum_{m=1}^{M} \dot{P}_n^{(m)}(t) = \sum_{m=1}^{M} A_n \bar{\dot{\rho}}_n^{(m)}(t) = A_n \sum_{m=1}^{M} \bar{\dot{\rho}}_n^{(m)}(t) \tag{5}$$

Hence, according to the definitions 13, 14 and 15, the overall net population generation rate in a community is separated into different parts depending on the population Groups $S^{(m)}$ of the analysis.

DEFINITION 16. "RATE OF GENERATION OF ADVANCEMENT COMPONENT $\dot{G}^{(m)}_{n,j}$" is defined as the rate of extra component of advancement $G^{(m)}_{n,j}$ that the community C_n can supply, during a specified time period, to increase the particular component of average degree of individual advancement $\overline{D}^{(m)}_{n,j}$ of the inhabitants of the population group $S^{(m)}$, solely due to the conditions of the community C_n.

It is important to note that definition 16 strictly states that the generation rate is solely due to the community C_n; hence, the transfer of advancement resulting from the migratory motion between the communities must necessarily be excluded in the evaluation of the quantity.

DEFINITION 17. "NET GROUP MIGRATION $Q^{(m)}_{n,\ell}$" is defined as the total number of human beings of the population group $S^{(m)}$ who change their residencies from the community C_ℓ to the community C_n during a specified time period.

DEFINITION 18a. "AVERAGE GROUP MIGRATION IN-FLUX DENSITY $\overline{q}^{(m)}_{n,\ell}$" is defined as the average value of the number of inhabitants, belonging to population group $S^{(m)}$, who take residence on a unit area of the community C_n due to migration from the community C_ℓ during a specified time period.

DEFINITION 18b. "AVERAGE GROUP MIGRATION OUT-FLUX DENSITY $\overline{q}^{(m)}_{\ell,n}$" is defined as the average value of the number of inhabitants, belonging to population group $S^{(m)}$, who terminate their residencies on a unit area of the community C_n to migrate to another community C_ℓ during a specified time period.

As a consequence of the definitions 17, 18a and 18b, the net group migration $Q^{(m)}_{n,\ell}$ to the community C_n from the community C_ℓ can be expressed in terms of the migration flux densities as:

$$Q_{n,\ell}^{(m)}(t) = A_n \bar{q}_{n,\ell}^{(m)} \quad , \quad Q_{\ell,n}^{(m)} = A_n \bar{q}_{\ell,n}^{(m)} \tag{6}$$

The definitions and the postulates of this section complete the necessary mathematical preliminaries for the development of the mathematical model for the human habitation problems between communities. It is in order to summarize the fact that all the pertinent mathematical quantities were reduced to time dependencies; hence, the further development of the theory will only consider on the average behaviors of the phenomenon with respect to time without taking into consideration the local changes in the communities and the individual conditions of the inhabitants. However, it should also be emphasized that the simplicity of the model is not essential, and that the fundamental concepts of this theory can easily be extended to include at least the spatial variations in the communities.

CONSERVATION PRINCIPLES AND DERIVATION OF GOVERNING SYSTEM OF DIFFERENTIAL EQUATIONS

Associated with any natural and physical phenomenon, there exists certain conservation principles which need to be satisfied at all times regardless of the simplicity or complexity of the problem under consideration. The mathematical formulation of these principles, through the appropriate logical transformation models, leads to the establishment of systems of governing differential equations whose solutions, under specified boundary conditions, in turn represent the actual, observable behavior characteristics of the physical phenomenon. Since the dynamic behavior of human habitation, ekistics, is indeed a natural phenomenon, it is obviously conceivable to expect the existence of certain conservation principles.

To consider the phenomenon in most general form, it is assumed that there exists N number of distinct communities, C_1, $C_2, \ldots$, $C_n, \ldots$, C_N, which are continuously affected by the interactions among each other and with the possibly existing external environment C_e through the migratory motion of human beings between the communities. A schematic representation of the general phenomenon is given in Figure 1.

POSTULATE 3. When an individual migrates from a community C_ℓ by removing his residency, the contribution of his individual behavior characteristics to the overall conditions of the community and in turn the responsibility of the community to the individual are immediately terminated.

POSTULATE 4. When an individual migrates to a community C_n by establishing his residency, the contribution of his individual behavior characteristics to the overall conditions of the community and in turn the responsibility of the community to the individual are immediately initiated. According to postulates 3 and 4, the cases when an individual resides in one community and earns his living in another community must necessarily be discarded in the formulations. Although this condition may be considered as rather stringent in view of the possibility of existence of commuters, the communities of the general system can always be defined in such a manner that the number of these special cases becomes logically negligible in comparison to the overall complete migration phenomenon.

POSTULATE 5. When an individual migrates to a new community C_n, he automatically contributes the individual behavior characteristics that he possesses to the overall conditions of his new community.

POSTULATE 6. After the migration of an inhabitant to a new community C_n, any changes that occur in his individual behavior characteristics are solely due to the overall conditions of his new community C_n.

Postulates 5 and 6 represent the necessary and logical assumptions to determine the effects of, particularly, the degree of individual advancement of the migrating people on the advancement of the community; and con-

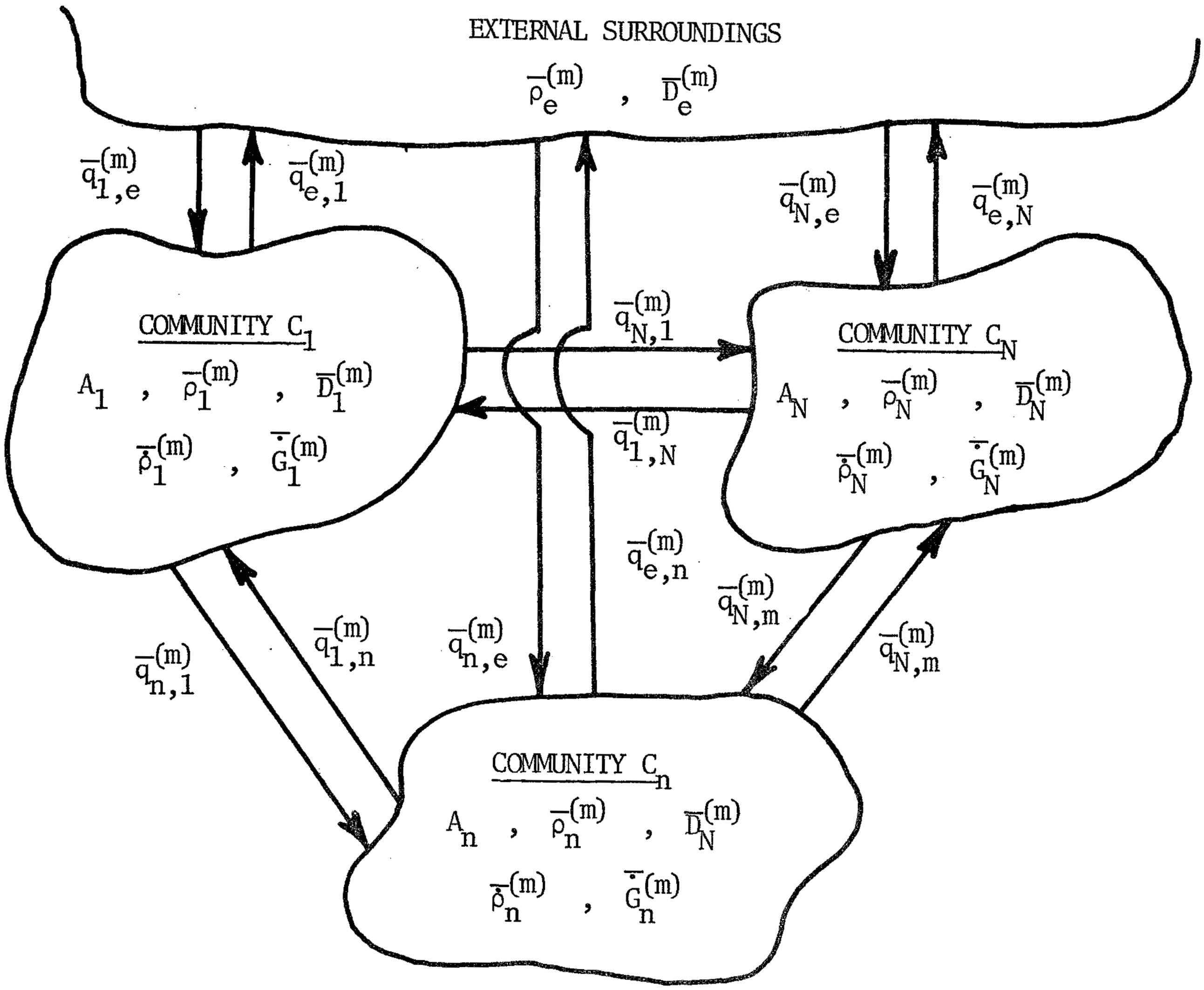

Figure 1. Schematic Representation of the General Dynamic Ekistics Problem.

versely the conditions of the community in changing the average degree of individual advancement of the inhabitants who have migrated to the community.

POSTULATE 7. On the average, the component of average degree of individual advancement $\bar{D}^{*(m)}_{\ell,j}$ of the inhabitants, belonging to the population group $S^{(m)}$, who migrate from the community C_ℓ to the community C_n can be represented as:

a. the value of the component of average degree of individual advancement $\bar{D}^{(m)}_{\ell,j}$ of the community C_ℓ,

b. the value of the component of average degree of individual advancement $\bar{D}^{(m)}_{n,j}$ of the community C_n,

c. the value which is proportional to either or both of the components of the average degree of individual advancement $\bar{D}^{(m)}_{\ell,j}$ and/or $\bar{D}^{(m)}_{n,j}$ of the communities C_ℓ and C_n, i.e.,

$$\underline{a}.\ \bar{D}^{*(m)}_{\ell,j} = \bar{D}^{(m)}_{\ell,j} \quad , \quad \underline{b}.\ \bar{D}^{*(m)}_{\ell,j} = \bar{D}^{(m)}_{n,j} \quad , \quad \underline{c}.\ \bar{D}^{*(m)}_{\ell,j} = \alpha_\ell \bar{D}^{(m)}_{\ell,j} + \alpha_m \bar{D}^{(m)}_{n,j}$$

It is important to realize that any one of the three parts of postulate 7 is a logical representation of the degree of the individual advancement of the human beings migrating from a community to another. The concrete determination of the correct part for the migration between two communities ultimately depends on the sociological and psychological characteristics of the inhabitants of the two communities.

The final set of postulates of this section completes the necessary preliminary assumptions for the statement and the application of the fundamental conservation principles for the derivation of the system of governing differential equations for the general problem.

Principle of Conservation of Group Population:

For any community C_n, the net rate of increase of any group population

$P_n^{(m)}$ is necessarily equal to the sum of the rate of generation of group population $\dot{P}_n^{(m)}$ in the community C_n and the overall net group migration $Q_{n,\ell}^{(m)}$ from the other communities C_ℓ into the community C_n.

The principle of conservation of group population leads to the following theorem.

THEOREM 1. In any community C_n, the instantaneous, time rate of change of any average population group density $\rho_n^{(m)}$ is given by:

$$\frac{d\bar{\rho}_n^{(m)}}{dt} = \bar{\dot{\rho}}_n^{(m)} + \sum_{\ell=1\neq n}^{N} [\bar{q}_{n,\ell}^{(m)} - \bar{q}_{\ell,n}^{(m)}] \tag{7}$$

Proof: Employing the principle of conservation of population in mathematical form,

$$\frac{dP_n^{(m)}}{dt} = \dot{P}_n^{(m)} + \sum_{\ell=1\neq n}^{N} Q_{n,\ell}^{(m)} - \sum_{\ell=1\neq n}^{N} Q_{\ell,n}^{(m)} \tag{8}$$

Substituting the average quantities from Eqs. (1), (4), (6) in Eq. (8),

$$\frac{d}{dt}[A_n\bar{\rho}_n^{(m)}] = A_n\bar{\dot{\rho}}_n^{(m)} + \sum_{\ell=1\neq n}^{N} A_n\bar{q}_{n,\ell}^{(m)} - \sum_{\ell=1\neq n}^{N} A_n\bar{q}_{\ell,n}^{(m)} \tag{9}$$

Noting that according to postulate 1, A_n is a constant and combining the summation terms,

$$\frac{d\bar{\rho}_n^{(m)}}{dt} = \bar{\dot{\rho}}_n^{(m)} + \sum_{\ell=1\neq n}^{N} [\bar{q}_{n,\ell}^{(m)} - \bar{q}_{\ell,n}^{(m)}] \tag{7}$$

which proves the theorem.

Principle of Conservation of Component of Advancement:

For any community C_n, the net rate of increase of the component of Advancement $G_{n,j}^{(m)}$ for the population group $S^{(m)}$ is necessarily equal to the sum of the rate of generation of advancement component $\dot{G}_{n,j}^{(m)}$ in the community C_n

and the overall transfer of group advancement due to the net migratory effects from the other communities C_ℓ into the community C_n.

The principle of conservation of component of advancement leads to the following theorem.

THEOREM 2. In any community C_n, the instantaneous, time rate of change of any component of average degree of individual advancement $\bar{D}_{n,j}^{(m)}$ is given by:

$$\frac{d\bar{D}_{n,j}^{(m)}}{dt} = \left[\frac{\dot{G}_{n,j}^{(m)}}{A_n\bar{\rho}_n^{(m)}} - \frac{\bar{D}_{n,j}^{(m)}\dot{\bar{\rho}}_n^{(m)}}{\rho_n^{(m)}}\right] + \frac{1}{\rho_n^{(m)}} \sum_{\ell=1\neq n}^{N} \Big\{[\bar{D}_{\ell,j}^{*(m)} - \bar{D}_{n,j}^{(m)}]\bar{q}_{n,\ell}^{(m)}$$

$$- [\bar{D}_{n,j}^{*(m)} - \bar{D}_{n,j}^{(m)}]\bar{q}_{\ell,n}^{(m)}\Big\} \tag{10}$$

Proof: Employing the principle of conservation fo component of advancement in mathematical form,

$$\frac{dG_{n,j}^{(m)}}{dt} = \dot{G}_{n,j}^{(m)} + \sum_{\ell=1\neq n}^{N} \bar{D}_{\ell,j}^{*(m)}Q_{n,\ell}^{(m)} - \sum_{\ell=1\neq n}^{N} \bar{D}_{n,j}^{*(m)}Q_{\ell,n}^{(m)} \tag{11}$$

Substituting the average quantities from Eqs. (3), (6) in Eq. (11),

$$\frac{d}{dt}[A_n\bar{\rho}_n^{(m)}\bar{D}_{n,j}^{(m)}] = \dot{G}_{n,j}^{(m)} + \sum_{\ell=1\neq n}^{N} \bar{D}_{\ell,j}^{*(m)}A_n\bar{q}_{n,\ell}^{(m)} - \sum_{\ell=1\neq n}^{N} \bar{D}_{n,j}^{*(m)}A_n\bar{q}_{\ell,n}^{(m)} \tag{12}$$

Noting that according to postulate 1 A_n is a constant, expanding the term on the left and combining the summation terms,

$$\bar{D}_{n,j}^{(m)}\frac{d\bar{\rho}_n^{(m)}}{dt} + \bar{\rho}_n^{(m)}\frac{d\bar{D}_{n,j}^{(m)}}{dt} = \frac{\dot{G}_{n,j}^{(m)}}{A_n} + \sum_{\ell=1\neq n}^{N} [\bar{D}_{\ell,j}^{*(m)}\bar{q}_{n,\ell}^{(m)} - \bar{D}_{n,j}^{*(m)}\bar{q}_{\ell,n}^{(m)}] \tag{13}$$

Multiplying the result of theorem 1, Eq. (7) by $\bar{D}_{n,j}^{(m)}$, substituting in Eq. (13) to eliminate the first term on the left and rearranging,

$$\frac{d\bar{D}_{n,j}^{(m)}}{dt} = \left[\frac{\dot{G}_{n,j}^{(m)}}{A_n\bar{\rho}_n^{(m)}} - \frac{\bar{D}_{n,j}^{(m)}\bar{\dot{\rho}}_n^{(m)}}{\rho_n^{(m)}}\right] + \frac{1}{\rho_n^{(m)}} \sum_{\ell=1\neq n}^{N} \left\{[\bar{D}_{\ell,j}^{*(m)} - \bar{D}_{n,j}^{(m)}]\bar{q}_{n,\ell}^{(m)} - [\bar{D}_{n,j}^{*(m)} - \bar{D}_{n,j}^{(m)}]\bar{q}_{\ell,n}^{(m)}\right\} \tag{10}$$

which proves the theorem.

The set of (NxM) number of equations resulting from theorem 1 Eq. (17) is self explanatory in view of the principle of conservation of group population; and indeed the system can be considered as an extension of the proposed theory of Refs. 19-23. However, the second set of (NxMxJ) equations represent the contribution of the new mathematical theory, Ref. 33; and the system contains two characteristics of significant importance which warrant individual explanation in relation to the natural phenomenon of population ekistics.

The first bracketed term on the right of Eq. (10) represents the net available amount of generation of advancement component which can be used to increase the component of average degree of individual advancement $\bar{D}_{n,j}^{(m)}$ of the population group $S^{(m)}$ in the community C_n. The most significant property of the particular term is the existence of $\bar{D}_{n,j}^{(m)}\bar{\dot{\rho}}_n^{(m)}/\rho_n^{(m)}$ which represents the effect of the population generation rate on the average living standard. The term clearly establishes that if the average rate of generation of population group density is positive, $\bar{\dot{\rho}}_n^{(m)} > 0$, i.e., contributing to the increase of the group population density $\bar{\rho}_n^{(m)}$, then, a certain amount of advancement opportunity for the particular population group $S^{(m)}$ of the community C_n is lost due to the population generation. Furthermore, since the generation rate is multiplied by the component of average degree of individual advancement, the loss of advancement opportunity is directly proportional to the average living standard of the population group $S^{(m)}$ in the community C_n. This result of the mathematical theory is of ultimate significant importance since it clearly illustrates that the population explosion is considerably more detrimental to the well being of the individual in an affluent society than in an underdeveloped one.

The summation term of the right of Eq. (10) illustrates the overall effect of the migration to the average living standards of the inhabitants in a community. The first bracketed term in the summation represents the contribution of the people, belonging to population group $S^{(m)}$, who migrate from other communities C_ℓ into community C_n. In view of the coefficient $[\bar{D}^{*(m)}_{\ell,j} - \bar{D}^{(m)}_{n,j}]$ of the $\bar{q}^{(m)}_{n,\ell}$, it is immediately evident that the average living conditions of the particular population group $S^{(m)}$ will improve if the migrating people to the community have component of average degree of individual advancement $\bar{D}^{*(m)}_{\ell,j}$ higher than the already established value $\bar{D}^{(m)}_{n,j}$ in the community. The second bracketed term represents the similar argument that the average living standards of a population group $S^{(m)}$ will increase if the migrating people out of the community have component of average degree of individual advancement $\bar{D}^{*(m)}_{n,j}$ less than the already established value $\bar{D}^{(m)}_{n,j}$ in the community. Hence, the mathematical theory clearly illustrates the importance of the living standards of the migrating people to the overall living standards of the community.

The mathematical system comprising of the two sets of equations, Eqs. (7) and (10), represents (D+1)xNxM number of first order nonlinear differential equations in unknowns $\bar{\rho}^{(m)}_n$(NxM), $\bar{D}^{(m)}_{n,j}$(NxMxJ), $\bar{D}^{*(m)}_{n,j}$(NxMxJ), $\bar{D}^{*(m)}_{\ell,j}$(NxMxJ), $\bar{\rho}^{(m)}_n$(NxM), $\dot{G}^{(m)}_{n,j}$(NxMxJ), $\bar{q}^{(m)}_{n,\ell}$[NxMx(N-1)], $\bar{q}^{(m)}_{\ell,n}$[NxMx(N-1)], i.e., 2x(N+2J)xNxM number of unknowns; hence, it is highly inconsistent for a unique set of solutions.

Since the results of the theorems 1 and 2, Eqs. (7) and (10) are the mathematical representations of the only two necessary and available conservation principles for the solutions of the two sets of principal unknowns $\bar{\rho}_n^{(m)}$ (NxM) and $\bar{D}_{n,j}^{(m)}$ (JxNxM) of the problem, the remaining unknowns of secondary importance, $\bar{D}_{n,j}^{*(m)}$, $\bar{D}_{\ell,j}^{*(m)}$, $\dot{\bar{\rho}}_n^{(m)}$, $\dot{G}_{n,j}^{(m)}$, $\bar{q}_{n,\ell}^{(m)}$ and $\bar{q}_{\ell,n}^{(m)}$, need to be eliminated from the system by establishing certain additional mathematical relations representing the observable behavior characteristics of the inhabitants of the communities in the general system.

LAW OF MIGRATORY MOTION

The task of investigating the phenomenon of migratory motion between the communities under the combined effects of the environmental conditions of the communities, may seem to be a problem of such immense magnitude of complexity that one may be tempted to conclude that any quantitative mathematical formulation is strictly impossible. However, discarding the unpredictable behavior characteristics of a small number of human beings, on the average, by formal argumentative reasoning, one may easily accept the existence of three independent, basic and in a sense quite general reasons which influence the migratory motions between the communities. Then the problem reduces to independent, quantitative formulation of these motivations and, ultimately, to establish a relation which encompasses the totality of the phenomena which stimulate the migratory motion.

a. Motivation of Population Density Potential

In various studies in mathematical biophysics by Skellam [19], Kerner [25-29], Beckmann [33] and Landahl [20,21], it is well established that the spatial variations in population densities of biological species result in migratory motions which can be expressed mathematically as proportionality relations to the gradients of population densities.

Accepting the argument as valid, and indeed plausible, the gradient type dependencies can be modified and extended by the inclusion of the influences of the population differences of all population groups on the migratory motion of each population group $S^{(m)}$ from the community C_ℓ to the community C_n. Hence, a general migration law for the influences of population group density potentials can be stated as follows.

<u>Law of Migratory Motion Due to Population Potentials</u>: If there exist differences in the average population group densities between two communities, then the average group migration flux density $\bar{q}^{(m)}_{n,\ell}$ of the population group $S^{(m)}$ from the community C_ℓ to community C_n depends linearly on the average population group density $\bar{\rho}^{(m)}_\ell$ of the population group $S^{(m)}$ in the community C_ℓ with the proportionality relation depending on the sum of the power type dependencies of the potentials of all average population group densities $\bar{\rho}^{(k)}_\ell$ and $\bar{\rho}^{(k)}_n$ between the two communities C_n and C_ℓ, i.e.,

$$\bar{q}^{(m)}_{n,\ell}(\bar{\rho}) = \left\{ \sum_{k=1}^{M} \alpha^{(k)}_{n,\ell} [\bar{\rho}^{(k)}_n - \bar{\rho}^{(k)}_\ell]^{p_{k,n,\ell}} \right\} \bar{\rho}^{(m)}_\ell \tag{14}$$

The parameters $\alpha^{(k)}_{n,\ell}$ and $p_{k,n,\ell}$ in general may depend on the principal unknowns of the system as:

$$\begin{aligned} \alpha^{(k)}_{n,\ell} &= \alpha^{(k)}_{n,} [\bar{\rho}^{(k)}_n, \bar{D}^{(k)}_{n,j}, \bar{\rho}^{(k)}_\ell, D^{(k)}_{\ell,j}] \geq 0 \\ p_{k,n,\ell} &= p_{k,n,\ell} [\bar{\rho}^{(k)}_n, \bar{D}^{(k)}_{n,j}, \bar{\rho}^{(k)}_\ell, \bar{D}^{(k)}_{\ell,j}] \geq 0 \end{aligned} \tag{15}$$

The evaluation of the parameters $\alpha^{(k)}_{n,\ell}$ and $p_{k,n,\ell}$ strictly depends on the observable, statistical data about the sociological and psychological attitudes of the inhabitants of the two communities in relation to their tendencies to migrate. Considering the fact that the law of migratory motion due to population potentials depends in general on the average population densities and the average degrees of individual advancement, the parameters $\alpha_{n,\ell}^{(k)}$ and $p_{k,n,\ell}$ evaluated statistically for the interaction between two communities C_n and C_ℓ can be used as approximate universal values for all the communities since they depend solely on the attitudes of the inhabitants which can be considered rather universal in the overall social structure of the system.

b. Motivation of Advancement Potential

The potential laws based on the population density gradients were employed previously for formulating the migration flux phenomena associated with lower evolutionary level biological species [Refs. 19,20,21,23,25-29]. However, the fundamental motivation for the migration of a human being from a community C_ℓ to another community C_n is not in general based on population density differences; but rather it is based on the existence of the possibility to improve his living conditions in the latter community.

The only logical way he can decide that such an improvement is indeed possible is to compare the levels of the average degree of individual advancements in the two communities. If such a comparison results in favor of his community, then, solely from the point of view of improving his living conditions, he will retain his residency in his community. However, if the community C_n has a higher level of average degree of individual advancement in comparison to his own community, then, he has a logical motivation to move his residency to community C_n. Consequently, the average group migration flux density $\bar{q}_{n,\ell}^{(m)}$ will depend on the difference, i.e., the potential, between the components of average degree of individual advancement $\bar{D}_{n,j}^{(m)}$ and $\bar{D}_{\ell,j}^{(m)}$ of the two communities C_n and C_ℓ.

Furthermore, since the migration is from community C_ℓ, it is logical to assume that a certain amount of the population in community C_ℓ, indeed in proportion to the population, will migrate to the new community C_n.

Hence, a quantitative mathematical formulation for the effects of the living conditions of the communities on the average group migration flux densities can be established according to the following statement.

Law of Migratory Motion Due to Advancement Potentials: If there exist differences in the components of average degree of individual advancement between two communities, then the average group migration flux density $\bar{q}^{(m)}_{n,\ell}$ of the population group $S^{(m)}$ from the community C_ℓ to the community C_n depends linearly on the average population group density $\bar{\rho}^{(m)}_\ell$ of the population group $S^{(m)}$ in the community C_ℓ with the proportionality relation depending on the total sum of the power type dependencies of the potentials of all the components of average degree of individual advancement $\bar{D}^{(m)}_{n,j}$ and $\bar{D}^{(m)}_{\ell,j}$ of all population groups $S^{(m)}$ between the two communities C_n and C_ℓ, i.e.,

$$\bar{q}^{(m)}_{n,\ell}(\bar{D}) = \left\{\sum_{k=1}^{N} \sum_{j=1}^{J} \beta^{(k)}_{n,\ell,j}[\bar{D}^{(k)}_{n,j} - \bar{D}^{(k)}_{\ell,j}]^{r_{k,n,\ell,j}}\right\}\bar{\rho}^{(m)}_\ell \tag{16}$$

The parameters $\beta^{(k)}_{n,\ell,j}$ and $r_{k,n,\ell,j}$ in general may depend on the principal unknowns of the system as:

$$\begin{aligned} \beta^{(k)}_{n,\ell,j} &= \beta^{(k)}_{n,\ell,j}[\bar{\rho}^{(k)}_n, \bar{D}^{(k)}_{n,j}, \bar{\rho}^{(k)}_\ell, \bar{D}_{\ell,j}] \geq 0 \\ r_{k,n,\ell,j} &= r_{k,n,\ell,j}[\bar{\rho}^{(k)}_n, \bar{D}^{(k)}_{n,j}, \bar{\rho}^{(k)}_\ell, \bar{D}_{\ell,j}] \geq 0 \end{aligned} \tag{17}$$

The parameters $\beta^{(k)}_{n,\ell,j}$ and $r^{(k)}_{k,n,\ell,j}$, similar to the law for population potential, need to be determined by the statistical analysis of the attitudes of the inhabitants of the communities; and the values for the parameters could be considered as approximately universal in the analysis of system of communities.

It is of significant importance to realize that the expressions for the average group migration flux densities $\bar{q}^{(m)}_{n,\ell}$, Eqs. (14,16), also establish the method for the statistical determination of the parameters $\alpha^{(k)}_{n,\ell}$, $\beta^{(k)}_{n,\ell,j}$, $p_{k,n,\ell}$ and $r_{k,n,\ell,j}$. Indeed, each parameter can be determined independently from the remaining ones by considering statistically observed data about the existing values of the associated potential and the amount of migration flux density solely due to the effect of the particular potential. Consequently, the establishment of the two migratory motion laws also lead to the simplification of the statistical analysis for the compilation of the necessary data about the behavior characteristics of the inhabitants.

c. Potentially Unmotivated Migratory Effects

In addition to the two fundamental motivations for the migration of inhabitants, there may exist certain other reasons not related to either the advancement or the population potentials between the communities. In general the cumulative effect of these additional migratory phenomena from community C_ℓ to community C_n depends on the living conditions of both communities and may be expressed mathematically as an average group migration flux density $\bar{q}^{*(m)}_{n,\ell}$,

$$\bar{q}^{*(m)}_{n,\ell} = \bar{q}^{*(m)}_{n,\ell}[\bar{\rho}^{(m)}_{n}, \bar{\rho}^{(m)}_{\ell}, \bar{D}^{(m)}_{n}, \bar{D}^{(m)}_{\ell}] \tag{18}$$

for most communities it is logical to assume that this particular form of migration is negligible compared to the fundamental migration fluxes due to the advancement and population potentials, hence it can be excluded in the analysis. However, if it becomes established that the effect is of significant value, then, since the definite reasons for the migration would be known, it can easily be determined by statistical observations to include its quantitative influence in the analysis of particular problems.

Considering that the net migratory phenomenon between two communities must necessarily be due to the overall, cumulative result of all the individual effects, the general law of migratory motion may be stated as follows.

The Law of Migratory Motion

The net average group migration flux density $\bar{q}_{n,\ell}^{(m)}$ of the population group $S^{(m)}$ from the community C_ℓ to the community C_n is the net sum of the average group migration flux density $\bar{q}_{n,\ell}^{*(m)}$, due to the unmotivated migration of the inhabitants, and the motivated average group migration flux densities $\bar{q}_{n,\ell}^{(m)}(\bar{\rho})$ and $\bar{q}_{n,\ell}^{(m)}(\bar{D})$ which result as consequences of the existence of potentials in the average population group densities $\bar{\rho}_n^{(m)}$ and $\bar{\rho}_\ell^{(m)}$ and the components of average degree of individual advancement $\bar{D}_{n,j}^{(m)}$ and $\bar{D}_{\ell,j}^{(m)}$ between the communities C_n and C_ℓ for the population group $S^{(m)}$, i.e.,

$$\bar{q}_{n,\ell}^{(m)} = \bar{q}_{n,\ell}^{*(m)} + \bar{q}_{n,\ell}^{(m)}(\bar{\rho}) + \bar{q}_{n,\ell}^{(m)}(\bar{D}) \tag{19}$$

where each contribution $\bar{q}_{n,\ell}^{*(m)}$, $\bar{q}_{n,\ell}^{(m)}(\bar{\rho})$ and $\bar{q}_{n,\ell}^{(m)}(\bar{D})$ to the overall migratory motions can be evaluated individually according to Eqs. (14), (16) and (18).

It is important to note that according to the original definition of the average group migration flux density $\bar{q}_{n,\ell}^{(m)}$, the quantity must necessarily be positive to be consistent with the mathematical formulation of the conservation principles. However, according to the mathematical form of the law of migratory motion, depending on the net sum of the values of the potential terms, the quantity could possibly attain negative values. From logical considerations, as discussed previously in relation to motivations, the negative values for the total potential must necessarily be excluded since they represent the nonrealistic conditions when the inhabitants of a community do not have any reason for migrating to another community. Consequently, the mathematical expression for the average migration flux density must be modified as:

$$\underline{a}.\quad \text{For}\quad \bar{q}_{n,\ell}^{(m)}(\rho) + \bar{q}_{n,\ell}^{(m)}(D) > 0 \;:\; \bar{q}_{n,\ell}^{(m)} = \bar{q}_{n,\ell}^{*(m)} + \bar{q}_{n,\ell}^{(m)}(\rho) + \bar{q}_{n,\ell}^{(m)}(D) \tag{20}$$

$$\underline{b}.\quad \text{For}\quad \bar{q}_{n,\ell}^{(m)}(\rho) + \bar{q}_{n,\ell}^{(m)}(D) \leq 0 \;:\; \bar{q}_{n,\ell}^{(m)} = \bar{q}_{n,\ell}^{*(m)} \tag{21}$$

Hence, equations (20) and (21) represent the definition of a continuous function $\bar{q}_{n,\ell}^{(m)}$ of variables $\bar{\rho}_n^{(m)}$, $\bar{D}_n^{(m)}$, $\bar{\rho}_\ell^{(m)}$ and $\bar{D}_\ell^{(m)}$ with sectionally continuous first order derivatives in the variables with the point of discontinuity of the derivative located according to the vanishing value of the sum of the potential terms. Since the function $\bar{q}_{n,\ell}^{(m)}$ itself is indeed continuous, this result does not create any mathematical difficulty in the general system of differential equations.

The law of migratory motion in its mathematically defined form as Eqs. (20) and (21) accomplishes the essential result of eliminating the unknown set of variables $\bar{q}_{n,\ell}^{(m)}$ of the original mathematical system in terms of $\bar{\rho}_n^{(m)}$ and $\bar{D}_{n,j}^{(m)}$ which are the selected principal unknown variables, average group population density and the component of average degree of individual advancement, respectively, for the communities C_n, in the formulation of the mathematical theory.

CONSTITUTIVE LAWS FOR THE MATHEMATICAL THEORY

With the elimination of average group migration flux density $\bar{q}_{n,\ell}^{(m)}$, the basic inconsistency difficulty of the mathematical system reduces to the representation of the average rate of generation of population group density $\bar{\dot{\rho}}_n^{(m)}$, the rate of generation of advancement component $\dot{G}_{n,j}^{(m)}$ and the component of average degree of individual advancement for migration $\bar{D}_{n,j}^{*(m)}$ of the system.

<u>The Law of Population Generation</u>: The average rate of generation of population group density $\bar{\dot{\rho}}_n^{(m)}$ for the population group $S^{(m)}$ in the community C_n depends directly on time t and the instantaneous conditions in the community C_n described by the average population group densitities $\bar{\rho}_n^{(k)}$ and the

components of average degree of individual advancement $\overline{D}_{n,j}^{(k)}$ of all the population groups $S^{(k)}$ in the community C_n, i.e.,

$$\bar{\dot{\rho}}_n^{(m)} = \bar{\dot{\rho}}_n^{(m)}[t,\bar{\rho}_n^{(k)},\overline{D}_{n,j}^{(k)}] \tag{22}$$

<u>The Law of Generation of Advancement:</u> The rate of generation of advancement component $\dot{G}_{n,j}^{(m)}$ for the population group $S^{(m)}$ in the community C_n depends directly on time t and the instantaneous conditions in the communities C_ℓ described by the average population group densities $\bar{\rho}_\ell^{(k)}$ and the components of average degree of individual advancement $\overline{D}_{\ell,j}^{(k)}$ of all the population groups $S^{(k)}$ in the communities C_ℓ, i.e.,

$$\dot{G}_{n,j}^{(m)} = \dot{G}_{n,j}^{(m)}[t,\bar{\rho}_\ell^{(k)},\overline{D}_{\ell,j}^{(k)}] \tag{23}$$

It is important to realize that the law of generation of advancement is considerably more general than the law of population generation since the advancement generation in a community C_n could easily depend on the living conditions in another community C_ℓ.

The significant importance of the two laws is the stated fact that they are both constitutive laws, i.e., they express the unknown variables $\bar{\dot{\rho}}_n^{(m)}$ and $\dot{G}_{n,j}^{(m)}$ as functional relations in terms of the principal unknown variables $\bar{\rho}_\ell^{(k)}$ and $\overline{D}_{\ell,j}^{(k)}$ without necessitating the differences as potentials for the variables.

The quantitative determination of the functional relations for the two constitutive laws again necessitate the statistical analysis of the sociological behavior characteristics of the inhabitants and the economical standards of the communities.

The ultimate result of the two constitutive laws establishes the elimination of the unknown quantities $\bar{\dot{\rho}}_n^{(m)}$ and $\dot{G}_{n,j}^{(m)}$ in terms of the principal variables t, $\bar{\rho}_\ell^{(k)}$ and $\overline{D}_{\ell,j}^{(k)}$ of the system; hence, it overcomes the second inconsistency difficulty associated with the mathematical formulation of the population ekistic problems.

Degree of Individual Advancement of Migrating Inhabitant

Considering the motivation of advancement potential as the sole reason for the migration, it would be logical to assume that the migrating inhabitants transport an average degree of individual advancement equivalent to the average degree of individual advancement of the community $C_{\ell}^{(m)}$ that they migrate from, expecting to improve their degree of advancement to the level of the average degree of individual advancement of the community C_n where they plan to establish their new residency. Conversely, if the motivation for migration was solely due to the population potential between the communities, then it would be logical to assume that the inhabitants that migrate to the new community have already established the level of the average degree of individual advancement of the community C_n where they plan to migrate. Consequently, one can conclude that the component of average degree of individual advancement $\overline{D}^{*(m)}_{\ell,j}$, which is transported by the average group migration flux density $\overline{q}^{(m)}_{n,\ell}$ of the population group $S^{(m)}$, depends directly on the components of average degree of individual advancement $\overline{D}^{(m)}_{n,j}$ and $\overline{D}^{(m)}_{\ell,j}$ in the two communities C_n and C_{ℓ}, i.e.,

$$\overline{D}^{*(m)}_{\ell,j} = \overline{D}^{*(m)}_{\ell,j}\,[\overline{D}^{(m)}_{\ell,j}, \overline{D}^{(m)}_{n,j}] \tag{24}$$

which is again a constitutive relation that can be determined by the appropriate statistical analyses of the behavior characteristics of the inhabitants in general.

CONSISTENT MATHEMATICAL SYSTEM

In view of the law of migratory motion and the constitutive laws, the unknowns of secondary importance, average group migration flux density $\overline{q}^{(m)}_{n,\ell}$, average rate of generation of group density $\dot{\bar{\rho}}_n^{(m)}$, rate of generation of advancement component $\dot{G}_{n,j}^{(m)}$ and the component of average degree of individual advancement of the migrating inhabitant $\bar{D}_{\ell,j}^{*(m)}$, can be eliminated from the original system in terms of principal unknowns of the formulation, average population group density $\bar{\rho}_n^{(m)}$ and the component of average degree of individual advancement $\bar{D}_{n,j}^{(m)}$.

Hence, the ultimate mathematical system, associated with the dynamic population ekistics problems, becomes a consistent set of [(1 + J)xNxM] number of governing, first order, nonlinear, ordinary differential equations in [(1 + J)xNxM] unknowns $\bar{\rho}_n^{(m)}$ and $\bar{D}_{n,j}^{(m)}$ as:

$$\frac{d\bar{\rho}_n^{(m)}}{dt} = U_n^{(m)}[t,\rho_n^{(m)},\bar{D}_{n,j}^{(m)}] \tag{25}$$

$$\frac{d\bar{D}_{n,j}^{(m)}}{dt} = V_{n,j}^{(m)}[t,\rho_n^{(m)},D_{n,j}^{(m)}] \tag{26}$$

$$m = 1, 2,\ldots, M \quad , \quad n = 1, 2,\ldots, N \quad , \quad j = 1, 2,\ldots, J$$

where the right hand side functions $U_n^{(m)}$ and $V_{n,j}^{(m)}$ in Eqs. (25) and (26) represent the abreviated forms of the right hand sides of the Eqs. (7) and (10), respectively.

Since the system, Eqs. (25) and (26) consists of [(1 + J)xNxM] number of dependent variables $\bar{\rho}_n^{(m)}$ and $\bar{D}_{n,j}^{(m)}$ as known values at a certain $t = t_o$, with $t_o = 0$ arbitrarily chosen without any loss of generality, i.e.,

$$\bar{\rho}_n^{(m)}(0) = \bar{\rho}_{no}^{(m)} \quad , \quad \bar{D}_{n,j}^{(m)}(0) = \bar{D}_{n,jo}^{(m)} \tag{27}$$

Hence, the mathematical system, Eqs. (25), (26) and (27) constitute a well paused, initial value problem which can possibly be solved for the unknowns $\bar{\rho}_n^{(m)}$ and $\bar{D}_{n,j}^{(m)}$.

Before attempting the solution of the system, it would be instructive to consider some of the significant analytical properties of the differential equations, Eqs. (25) and (26).

The set of differential equations, Eqs. (7) and (10) or Eqs. (25) and (26) and the set of initial conditions, Eq. (27) represent a nonlinear initial value problem for the unknown vector W(t) as:

$$\frac{dW}{dt}(t) = F[t,W(t)] \quad ; \quad W(0) = W_o \tag{28}$$

where the unknown vector W(t) represents the column vector of the unknown quantities $\bar{\rho}_n^{(m)}$ and $\bar{D}_{n,j}^{(m)}$ and the right hand side vector F[t,W(t)] represents the totality of the right hand side functions of Eqs. (7) and (10) as:

$$F_i[t,W(t)] = \dot{\bar{\rho}}_n^{(m)} + \sum_{\ell=1=n}^{N} [\bar{q}_{n,\ell}^{(m)} - \bar{q}_{\ell,n}^{(m)}] \quad , \quad 1 \leq i \leq (NxM) \tag{29}$$

$$F_i[t,W(t)] = [\frac{\dot{G}_{n,j}^{(m)}}{A_n\bar{\rho}_n^{(m)}} - \frac{\bar{D}_{n,j}^{(m)}\dot{\bar{\rho}}_n^{(m)}}{\bar{\rho}_n^{(m)}}] + \frac{1}{\bar{\rho}_n^{(m)}} \sum_{\ell=1-n}^{N} \Big\{ [\bar{D}_{\ell,j}^{*(m)} - \bar{D}_{n,j}^{(m)}]\bar{q}_{n,\ell}^{(m)}$$

$$- [\bar{D}_{n,j}^{*(m)} - \bar{D}_{n,j}^{(m)}]\bar{q}_{\ell,n}^{(m)} \Big\} \quad , \quad (MxN) < i \leq [(Jx1)xNxM] \tag{30}$$

If it is assumed that all the communities C_n contain inhabitants of all the population groups $S^{(m)}$ at all times, i.e., $\bar{\rho}_n^{(m)} > 0$ for all t, m, n, then the right hand side vector function F(t,W(t)] necessarily satisfies the Lipschitz conditions for the complete range of the independent variable t, $0 \leq t \leq \infty$; hence, from the well known theorems on differential equations, there exists a unique continuous vector solution W(t) for the system as a continuous extension of the initial conditions vector W_o.

An equilibrium (singular) point of the system, Eq. (28), is defined as the condition when the right hand side vector F[t,W(t)] becomes a null vector. As a consequence, the left hand side of the equation must also necessarily vanish, implying that at the point of equilibrium the time rate of change of all the variables attain null values as expected from the definition of equilibrium. Considering the elements of the vector, the condition $F[t,W(t)] = 0$ is equivalent to $[(J + 1)xNxM]$ number of algebraic equations of the form,

$$\bar{\dot{\rho}}_n^{(m)} + \sum_{\ell=1 \neq n}^{N} [\bar{q}_{n,\ell}^{(m)} - \bar{q}_{\ell,n}^{(m)}] = 0 \tag{31}$$

$$\frac{\dot{G}_{n,j}^{(m)}}{A_n} - \bar{D}_{n,j}^{(m)}\bar{\dot{\rho}}_n^{(m)} + \sum_{\ell=1 \neq n}^{N} \left\{ [\bar{D}_{\ell,j}^{*(m)} - \bar{D}_{n,j}^{(m)}]\bar{q}_{n,\ell}^{(m)} - [\bar{D}_{n,j}^{*(m)} - \bar{D}_{n,j}^{(m)}]\bar{q}_{\ell,n}^{(m)} \right\} = 0 \tag{32}$$

The first condition, Eq. (31), implies that for equilibrium conditions to exist in the system, the average rate of generation of population group density $\bar{\dot{\rho}}_n^{(m)}$ must necessarily be equal to the net average group migration flux density out of the community for all population groups $S^{(m)}$ and communities C_n.

Considering the fact that the system of communities is a closed system, i.e., the migration phenomena is restricted to the communities in the system, the sum of all average group migration flux densities over all communities n = 1, 2,..., N, necessarily vanishes. Hence, rearranging Eq. (31) and suming over n,

$$\sum_{n=1}^{N} \bar{\dot{\rho}}_n^{(m)} = - \sum_{n=1}^{N} \sum_{\ell=1 \neq n}^{N} [\bar{q}_{n,\ell}^{(m)} - \bar{q}_{\ell,n}^{(m)}] = 0$$

Therefore, Eq. (32), implies that equilibrium in the system is only possible if all average rates of generation of population density are not positive, i.e., there exists at least one $\bar{\dot{\rho}}_k^{(m)} < 0$.

The second condition, Eq. (32), can be rearranged, by substituting Eq. (31) in Eq. (32), as:

$$\frac{\dot{G}_{n,j}^{(m)}}{A_n} + \sum_{\ell=1 \neq n}^{N} [\bar{D}_{\ell,j}^{*(m)}\bar{q}_{n,\ell}^{(m)} - \bar{D}_{n,j}^{*(m)}\bar{q}_{\ell,n}^{(m)}] = 0 \tag{33}$$

which immediately implies that the rate of generation of advancement component per unit area of the community must necessarily compensate for the advancement loss due to the net value of the average group migration flux densities for all population groups $S^{(m)}$ in all communities C_n.

The detailed investigation of the stability of the solutions may require comprehensive treatment of the right hand side functions for particular problems and in general necessitate rigorous mathematical analysis of the linearized solutions of the system in the neighborhood of the equilibrium points. Consequently, the stability of the solutions cannot be guaranteed for the general mathematical system. However, considering the conditions when the right hand side functions are sufficiently bounded for the existence of the continuous solutions, from the general considerations of the Poincare-Liapunov stability theory, it would be indeed realistic to expect stable solutions for certain particular problems.

From the nonlinearity characteristics of the general system it is immediately evident that one cannot construct analytical solutions except for few rather restrictive problems. However, since it is established that unique solutions indeed exist for certain initial value problems, provided that the initial conditions do not constitute singular points, the set of first order ordinary differential equations can be readily integrated by the well known numerical techniques by employing high speed digital computers.

Hence, the general mathematical formulation of dynamic human habitation problems result in mathematical systems of differential equations which can be rendered consistent by the systematic application of statistical analysis to determine the necessary parameters in the potential and constitutive laws of the theory; and these systems can be solved by the application of modern day computers to obtain solutions which represent the quantitative behavior of the actual natural phenomena associated with all the communities in the system.

APPLICATION TO A PROBLEM OF TWO COMMUNITIES

In the course of investigation of a natural phenomenon, the ultimate judgement on the success or the failure of a proposed mathematical theory univocally depends on the feasibility of its application to specific problems of practical importance in describing certain phases of the general phenomenon. Consequently, it is necessary to illustrate the possibility of quantitative formulation of a specific problem, preferably of moderate complexity, to establish the validity of the mathematical theory for the dynamic human habitation problem between communities.

The problem to be considered is a hypothetical but naturally plausible one. The choice of a hypothetical case is definitely not of necessity for the application of the general theory but rather of brevity in illustrating the systematic transformation of the natural phenomenon into a well paused consistent mathematical system which can be quantitatively solved by some well known mathematical technique.

Statement of the Problem

Consider two small communities C_1 and C_2 and the external surroundings C_e under the simplifying assumption that the population groups in the communities can be combined into a single group for a preliminary analysis, i.e., $m = 1$,

$$\bar{\rho}_n^{(m)} \quad : \quad \bar{\rho}_1^{(1)} = \bar{\rho}_1 \quad , \quad \bar{\rho}_2^{(1)} = \bar{\rho}_2 \quad , \quad \bar{\rho}_e^{(1)} = \bar{\rho}_e \tag{34}$$

Furthermore, assume that the only pertinent component of the average degree of individual advancement can be represented by the average yearly income of the inhabitants, i.e., $j = 1$,

$$\bar{D}^{(m)}_{n,j} \quad : \quad \bar{D}^{(1)}_{1,1} = \bar{D}_1 \quad , \quad \bar{D}^{(1)}_{2,1} = \bar{D}_2 \quad , \quad \bar{D}^{(1)}_{e,1} = \bar{D}_e \tag{35}$$

Initially, the two communities C_1 and C_2 have approximately equal average population densities and the representative average yearly incomes per inhabitant in both communities are approximately the same. The two communities are initially under-populated and under-developed in comparison to the external surroundings, i.e.,

$$\bar{\rho}_{1o} = \bar{\rho}_{2o} < \bar{\rho}_{eo} \text{ and } \bar{D}_{1o} = D_{2o} < D_{eo} \tag{36}$$

Initially, the rates of generation of population in the communities are balanced out by the migration of the inhabitants from the communities to the external surroundings; hence, the average population densities of the communities are in equilibrium.

Furthermore, each community initially has a certain fixed gross yearly community income which is distributed among its inhabitants according to the average yearly income per inhabitant of the community.

Hence, if no changes are made in the communities, it is evident from the equilibrium conditions that both communities will remain under-populated and under-developed in comparison to the living standards of the external surroundings.

To develop one of the communities, e.g., C_1, a large industry, or a combination of various businesses, is established such that the gross yearly community income of the community C_1 is suddenly increased to a considerably larger value than at the initial under developed conditions of the community.

The problem is to quantitatively predict the living conditions in both communities C_1 and C_2 in the future due to the sudden creation of the large scale financial opportunity in community C_1.

Mathematical Formulation of the Problem

The governing equations for the particular problem can be obtained directly from the general system, Eqs. (7) and (10), by taking $m = 1$, $n = 1,2$, $\ell = e,1,2$ as:

$$\frac{d\bar{\rho}_1}{dt} = \bar{\dot{\rho}}_1 + \bar{q}_{1,e} - \bar{q}_{e,1} + \bar{q}_{1,2} - \bar{q}_{2,1} \tag{37}$$

$$\frac{d\bar{\rho}_2}{dt} = \bar{\dot{\rho}}_2 + \bar{q}_{2,e} - \bar{q}_{e,2} + \bar{q}_{2,1} - \bar{q}_{1,2} \tag{38}$$

$$\frac{d\bar{D}_1}{dt} = \frac{1}{\bar{\rho}_1}\Big\{(\dot{G}_1/A_1) - \bar{D}_1\bar{\rho}_1 + [\bar{D}_e^* - \bar{D}_1]\bar{q}_{2,1}$$
$$+ [\bar{D}_2^* - \bar{D}_1]\bar{q}_{1,2} + [\bar{D}_1 - \bar{D}_1^*]\bar{q}_{e,1} + [\bar{D}_1 - \bar{D}_1^*]\bar{q}_{2,1}\Big\} \tag{39}$$

$$\frac{d\bar{D}_2}{dt} = \frac{1}{\bar{\rho}_2}\Big\{(\dot{G}_2/A_2) - \bar{D}_2\bar{\rho}_2 + [\bar{D}_e^* - D_2]\bar{q}_{2,e}$$
$$+ [\bar{D}_1^* - \bar{D}_2]\bar{q}_{2,1} + [\bar{D}_2 - \bar{D}_2^*]\bar{q}_{e,2} + [\bar{D}_2 - \bar{D}_2^*]\bar{q}_{1,2}\Big\} \tag{40}$$

where for simplifying the notation, the superscripts were eliminated for the quantities,

$$\bar{\dot{\rho}}_n^{(m)} \quad : \quad \bar{\dot{\rho}}_1^{(1)} = \bar{\dot{\rho}}_1 \quad , \quad \bar{\dot{\rho}}_2^{(1)} = \bar{\dot{\rho}}_2$$

$$\bar{q}_{n,\ell}^{(m)} \quad : \quad \bar{q}_{1,2}^{(1)} = \bar{q}_{1,2} \quad , \quad \bar{q}_{2,1}^{(1)} = \bar{q}_{2,1} \quad , \quad \bar{q}_{1,e}^{(1)} = \bar{q}_{1,e} \quad , \quad \bar{q}_{e,1}^{(1)} = \bar{q}_{e,1} \quad ,$$
$$\bar{q}_{2,e}^{(1)} = \bar{q}_{2,e} \quad , \quad \bar{q}_{e,2} = \bar{q}_{e,2} \tag{41}$$
$$\bar{D}_{n,j}^{*(m)} \quad : \quad \bar{D}_{1,1}^{*(1)} = \bar{D}_1^* \quad , \quad \bar{D}_{2,1}^{*(1)} = \bar{D}_2^* \quad , \quad \bar{D}_{e,1}^{*(1)} = \bar{D}_e^*$$
$$\dot{G}_{n,j}^{(m)} \quad : \quad \dot{G}_{1,1}^{(1)} = \dot{G}_1 \quad , \quad \dot{G}_{2,1}^{(1)} = \dot{G}_2$$

The initial conditions at $t = 0$ are self-evident,

$$\bar{\rho}_1(0) = \bar{\rho}_{1o} \quad , \quad \bar{\rho}_2(0) = \bar{\rho}_{2o} \quad , \quad \bar{D}_1(0) = \bar{D}_{1o} \quad , \quad \bar{D}_2(0) = \bar{D}_{2o} \tag{42}$$

To determine the necessary relations, according to the potential and the constitutive laws, it is necessary to investigate the statistically observable characteristics of the behaviors of the inhabitants of the communities.

a. Statistically Observable Characteristics for External Environment C_e:

Condition a1. The average population density of the external surroundings remains constant for all time t, i.e.,

$$\bar{\rho}_e(t) = \bar{\rho}_e = \bar{\rho}_{eo} = \text{constant} \tag{43}$$

Condition a2. The average degree of individual advancement of the external surroundings remains constant for all time t, i.e.,

$$\bar{D}_e(t) = \bar{D}_e = \bar{D}_{eo} = \text{constant} \tag{44}$$

Condition a3. The inhabitants that migrate from the external surroundings have a degree of individual advancement equal to the average degree of individual advancement of the external surroundings, i.e.,

$$\bar{D}_e^* = \bar{D}_e = \bar{D}_{eo} = \text{constant} \tag{45}$$

b. Statistically Observable Characteristics for Community C_1:

Condition b1. The inhabitants that migrate from the community C_1 have a degree of individual advancement equal to the average degree of individual advancement of the community C_1, i.e.,

$$\bar{D}_1^* = \bar{D}_1 \tag{46}$$

Condition b2. The average rate of generation of population density $\dot{\bar{\rho}}_1$ in the community C_1 is proportional to the average population density $\bar{\rho}_1$ and it is inversely proportional to the average degree of individual advancement $\bar{D}_1$, i.e.,

$$\dot{\bar{\rho}}_1 = \frac{\Omega_1^* \bar{\rho}_1}{[1 + \gamma_1^* \bar{D}_1]} \tag{47}$$

where Ω_1^* and γ_1^* statistically determined positive constants.

<u>Condition b3.</u> The characteristic bounds for the degree of individual advancement are selected as $D_{min} = 0$ and the maximum bound D_{max} based on a sufficiently high yearly income G^* such that the average degree of individual advancement of the inhabitant is directly proportional to the average yearly income $\overline{E}_1$ as:

$$\overline{D}_1 = \overline{E}_1/G^* \quad , \quad \overline{E}_1 = G^*\overline{D}_1 \tag{48}$$

<u>Condition b4.</u> If the totality of the available business establishments in the community C_1 can afford to pay $\dot{B}_1^*$ in yearly expenditures to the inhabitants of the community, the difference $\Delta\dot{G}_1^*$ between $\dot{B}_1^*$ and the net amount of money that the inhabitants require as yearly income is the principal factor which determines the changes in the living conditions of the community, i.e.,

$$\Delta\dot{G}_1^* = \dot{B}_1^* - P_1\overline{E}_1 = \dot{B}_1^* - A_1\overline{\rho}_1 G^*\overline{D}_1 \tag{49}$$

<u>Condition b5.</u> If $\Delta\dot{G}_1^*$ is positive, i.e., there exists the financial possibility for the development of the community, then the total rate of generation of advancement for the community is proportional to the difference of the average degree of individual advancement of the community from its associated maximum bound, i.e.,

$$\dot{G}_1 = \kappa_1 \frac{(1 - \overline{D}_1)}{G^*} \Delta\dot{G}_1^* = \kappa_1 \frac{(1 - \overline{D}_1)}{G^*}[\dot{B}_1^* - A_1\overline{\rho}_1 G^*\overline{D}_1] \tag{50}$$

Hence, the average rate of generation of advancement per unit area, $\overline{g}_1$, may be defined as:

$$\overline{g}_1 = \frac{\dot{G}_1}{A_1} = \kappa_1 \frac{(1 - \overline{D}_1)}{G^*}[\overline{\dot{b}}_1^* - \overline{\rho}_1 G^*\overline{D}_1] \tag{51}$$

where: κ_1 is a positive constant and $\overline{\dot{b}}_1^* = \dot{B}_1^*/A_1$.

<u>Condition b6.</u> If $\Delta\dot{G}_1^*$ is negative, i.e., the community cannot support

its inhabitants at their level of the average degree of individual advancement, then the total rate of generation of advancement for the community is negative and it is directly proportional to the amount of deficiency in funds to support the total yearly income of the inhabitants, i.e.,

$$\bar{g}_1 = \frac{1}{G^*}[\bar{\dot{b}}_1^* - \bar{\rho}_1 G^* \bar{D}_1] \tag{52}$$

Conditions 5b and 6b can be expressed as a discontinuous function,

$$\bar{\dot{g}}_1 = \begin{cases} \kappa_1 \dfrac{(1 - D_1)}{G^*}[\bar{\dot{b}}_1^* - \bar{\rho}_1 G^* \bar{D}_1] & , \quad \bar{g}_1 > 0 \\ \dfrac{1}{G^*}[\bar{\dot{b}}_1^* - \bar{\rho}_1 G^* \bar{D}_1] & , \quad \bar{g} \leq 0 \end{cases} \tag{53}$$

c. Statistically Observable Characteristics for Community C_2:

Considering the similarities between the two communities, it can be formally assumed that the conditions 1 through 6 for the community C_1 also apply for the community $C_2^{(m)}$ with appropriate alterations in the mathematical representations of the conditions. Consequently, it would be sufficient to present the necessary results associated with the conditions without repeating the statements.

Conditions c1 and c2.

$$\bar{D}_2^* = \bar{D}_2 \quad , \quad \bar{\rho}_2 = \frac{\overset{*}{\Omega}_2 \bar{\rho}_2}{[1 + \gamma_2 \overset{*}{\bar{D}}_2]} \tag{54}$$

Conditions c3, c4, c5 and c6.

$$\bar{g}_2 = \begin{cases} \kappa_2 \dfrac{(1 - D_2)}{G^*}[\bar{\dot{b}}_2^* - \bar{\rho}_2 G^* \bar{D}_2] & , \quad \bar{g}_2 > 0 \\ \dfrac{1}{G^*}[\bar{\dot{b}}_2^* - \bar{\rho}_2 G^* \bar{D}_2] & , \quad \bar{g}_2 \leq 0 \end{cases} \tag{55}$$

The conditions b1 through c6, Eqs. (46) - (55), complete the necessary set of constitutive relations for the communities.

d. Statistically Observable Interaction Characteristics Between the Communities for Migration Phenomenon:

Condition d1. The average migration flux density due to the motivation of population potential $\overline{q}_{n,\ell}(\rho)$, from community C_ℓ to community C_n, is linearly proportional to the percentage of the existing difference between the levels of the average population density of the communities, i.e., $p_{k,n,\ell} = 1$, $m = k = 1$,

$$\overline{q}_{n,\ell}(\rho) = \overset{*}{\alpha}_{n,\ell}[1 - \frac{\overline{\rho}_n}{\overline{\rho}_\ell}]\overline{\rho}_\ell \tag{56}$$

Condition d2. The average migration flux density due to the motivation of advancement potential $\overline{q}_{n,\ell}(\overline{D})$ from community C_ℓ to community C_n is linearly proportional to the percentage of the existing difference between the levels of the average degree of individual advancement of the communities, i.e., $r_{k,n,\ell,j} = 1$, $m = k = j = 1$,

$$\overline{q}_{n,\ell}(\overline{D}) = \overset{*}{\beta}_{n,\ell}[\frac{\overline{D}_n}{\overline{D}_\ell} - 1]\overline{\rho}_\ell \tag{57}$$

Condition d3. The potentially unmotivated migratory effects between the communities are of negligible magnitude in comparison to the motivated migration phenomenon, i.e.,

$$\overline{q}_{n,\ell} = 0 \tag{58}$$

Combining the conditions d1, d2 and d3, the quantitative formulations of the migration flux densities between the communities become,

$$\overline{q}_{e,1} = \left\{\overset{*}{\alpha}_{e,1}[1 - \frac{\overline{\rho}_e}{\overline{\rho}_1}] + \overset{*}{\beta}_{e,1}[\frac{\overline{D}_e}{\overline{D}_1} - 1]\right\}\overline{\rho}_1$$

$$\overline{q}_{1,e} = \left\{\alpha^*_{1,e}[1 - \frac{\overline{\rho}_1}{\overline{\rho}_e}] + \beta^*_{1,e}[\frac{\overline{D}_1}{\overline{D}_e} - 1]\right\}\overline{\rho}_e$$

$$\overline{q}_{e,2} = \left\{\alpha^*_{1,e}[1 - \frac{\overline{\rho}_1}{\overline{\rho}_2}] + \beta^*_{e,2}[\frac{\overline{D}_e}{\overline{D}_2} - 1]\right\}\overline{\rho}_2$$

$$\overline{q}_{2,e} = \left\{\alpha^*_{2,e}[1 - \frac{\overline{\rho}_2}{\overline{\rho}_e}] + \beta^*_{2,e}[\frac{\overline{D}_2}{\overline{D}_e} - 1]\right\}\overline{\rho}_e \qquad (59)$$

$$\overline{q}_{2,1} = \left\{\alpha^*_{2,1}[1 - \frac{\overline{\rho}_2}{\overline{\rho}_1}] + \beta^*_{2,1}[\frac{\overline{D}_2}{\overline{D}_1} - 1]\right\}\overline{\rho}_1$$

$$\overline{q}_{1,2} = \left\{\alpha^*_{1,2}[1 - \frac{\overline{\rho}_1}{\overline{\rho}_2}] + \beta^*_{1,2}[\frac{\overline{D}_1}{\overline{D}_2} - 1]\right\}\overline{\rho}_2$$

with the associated discontinuity conditions,

$$\overline{q}_{n,\ell} = \begin{cases} \text{Equation (59)} & , \quad \overline{q}_{n,\ell} \geq 0 \\ 0 & , \quad \overline{q}_{n,\ell} < 0 \end{cases} \qquad (60)$$

The constant coefficients of the constitutive relations for the generation rates $\overline{\dot{\rho}}_n$ and $\overline{g}_n$, Eqs. (47), (53), (54), (55), and the potential relations for the migration flux densities $\overline{q}_{n,\ell}$, Eq. (59), can be determined from statistical considerations; hence, the consistent mathematical system for the four unknowns $\overline{\rho}_1$, $\overline{\rho}_2$, $\overline{D}_1$, $\overline{D}_2$ consists of four ordinary, nonlinear differential equations, Eqs. (37) - (40), the initial conditions, Eq. (42), and the constitutive and potential relations, Eqs. (47), (53), (54), (55) and (59), which can be integrated from the initial conditions for any set of numerically specified values of the constant coefficients $\alpha^*_{n,\ell}$, $\beta^*_{n,\ell}$, Ω^*_n, γ^*_n, κ_n, G^* and the constant values $\overline{\rho}_e$ and $\overline{D}_e$, provided that the system remains non-singular for the specified numerical values.

A Numerical Example for the Problem

The mathematical system for the two initially under-developed communities was applied to a numerical problem with the hypothetical values of the parameters associated with the statistically observable conditions about the communities.

Conditions a1, a2, a3.

$$\rho_e = 600 \text{ inhabitants/kilometer}^2$$
$$D_e = 0.6 \text{ degrees/inhabitant} \tag{61}$$

Conditions b2, b3, b4, b5, b6.

$$\Omega_1^* = 0.02/\text{year}$$
$$\gamma_1^* = 5 \text{ inhabitants/degree}$$
$$G^* = \$20{,}000/\text{degree year}$$
$$\theta_1 = 1/\text{year} \tag{62}$$
$$\bar{\dot{b}}_1^* = \$808{,}000/\text{kilometer}^2 \text{ year}, \; t < 0$$
$$\bar{\dot{b}}_1^* = \$4{,}000{,}000/\text{kilometer}^2 \text{ year}, \; t \geq 0$$

Conditions c2, c3, c4, c5, c6.

$$\Omega_2^* = 0.02/\text{year}$$
$$\gamma_2^* = 5 \text{ inhabitants/degree}$$
$$\theta_2 = 1/\text{year} \tag{63}$$
$$\bar{\dot{b}}_2^* = \$808{,}000/\text{kilometer}^2 \text{ year}, \; t \geq 0$$

Condition d1

$$\beta^*_{1,e} = \beta^*_{2,e} = 0.05/\text{year}$$
$$\beta^*_{e,1} = \beta^*_{e,2} = \beta^*_{2,1} = \beta^*_{1,2} = 0.005/\text{year} \tag{64}$$

It is important to note that the values of the parameters $\beta^*_{n,\ell}$ are chosen such that the conditions point out the hesitant behavior of the inhabitants of the small communities in migrating to another community in comparison to the inhabitants of the external surroundings.

Condition d2.

$$\alpha^*_{n,\ell} = 0 \quad , \quad n,\ell = e,1,2 \tag{65}$$

This condition implies that the migration phenomenon is not dependent on the population potential.

Condition d3.

$$\bar{q}^*_{n,\ell} = 0 \tag{66}$$

The initial conditions for the problem are specified as:

$$\begin{aligned}
\bar{\rho}_1(0) &= \bar{\rho}_{1o} = 200 \text{ inhabitants/kilometer}^2 \\
\bar{\rho}_2(0) &= \bar{\rho}_{2o} = 200 \text{ inhabitants/kilometer}^2 \\
\bar{D}_1(0) &= \bar{D}_{1o} = 0.2 \text{ degrees/inhabitant} \\
\bar{D}_2(0) &= \bar{D}_{2o} = 0.2 \text{ degrees/inhabitant}
\end{aligned} \tag{67}$$

The hypothetical statistical data was specified in such a way that the available income of \$808,000/kilometer^2 year in the two communities assures an equilibrium condition for the overall behavior of the two communities, i.e., both the average population densities $\bar{\rho}_1$, $\bar{\rho}_2$ and the average degrees of individual advancement $\bar{D}_1$, $\bar{D}_2$ remain constant if no further changes occur in the system.

To obtain the solutions for the living conditions of the communities after the increase of the available income in community C_1 to $\bar{b}_1^* = \$4,000,000/$ kilometer^2 year, the mathematical system of four nonlinear ordinary differential equations were numerically integrated by the fourth order Runge-Kutta method with Gill coefficients starting with the specified initial conditions.

The numerical results for the average population densities $\bar{\rho}_1$, $\bar{\rho}_2$ and the average degree of individual advancement $\bar{D}_1$, $\bar{D}_2$ are presented in Figure 2 for both communities.

As expected, the average population density $\bar{\rho}_1$ increases very rapidly

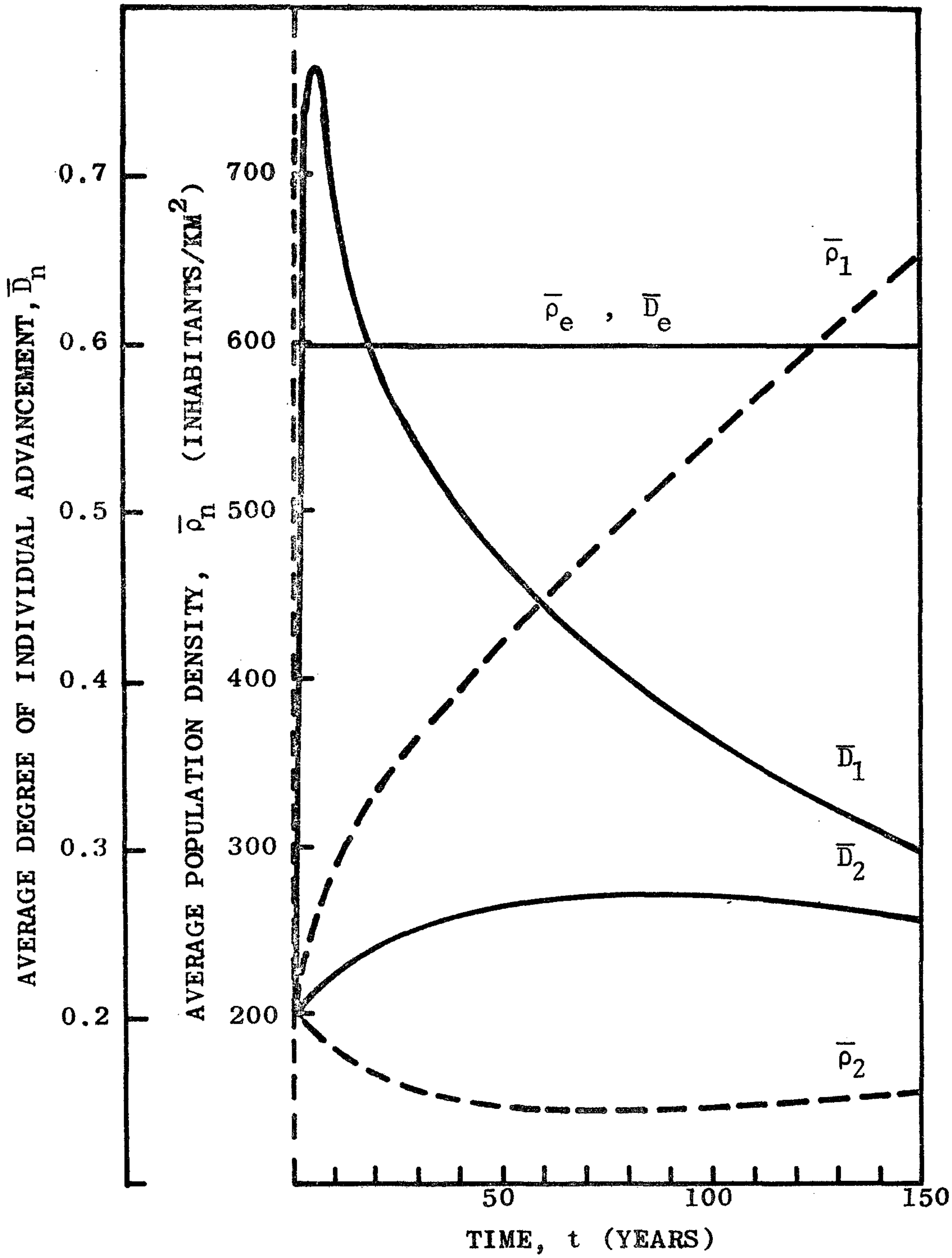

Figure 2. Variations of Average Population Density and Average Degree of Individual Advancement.

in community C_1 due to the migratory motion of the human beings from the external surroundings C_e and the second community C_2 to community C_1 in view of the establishment of a high level of average degree of individual advancement in the community C_1. Furthermore, since the average population density $\bar{\rho}_1$ increases, the average rate of generation of population increases in proportionality to the average population density, hence resulting in a substantial amount of population explosion due to the cumulative consequences of the two effects.

The average population density $\bar{\rho}_2$ in community C_2 undergoes simultaneous reduction due to the migration of the human beings from community C_2 to community C_1 in pursuit of possible advancement.

The significantly important consequence of the solutions is that the average degree of individual advancement $\bar{D}_1$, in community C_1, increases very rapidly in the first ten years due to the creation of the excess available wealth in the community. However, after fifteen years, the average population density $\bar{\rho}_1$ in the community attains the critical value at which the community can no longer support its inhabitants at their high level of average degree of individual advancement. Consequently, an immediate collapse of the average degree of individual advancement initiates and the living conditions deteriorate rapidly with the further over population of the community.

Contrary to the conditions in community C_1, the average degree of individual advancement $\bar{D}_2$ in community C_2 increases as the average population density $\bar{\rho}_2$ of the community decreases due to the migratory motion from community C_2 to community C_1. Considering that the community has a fixed available income $\bar{b}_2^* = \$808{,}000/\text{kilometer}^2$ year which is shared by its inhabitants; as the population decreases, the individual inhabitant of the community has a chance to earn more in a year, hence increasing the average degree of individual advancement D_2 of the community. Approximately after 100 years, corresponding to the time when the living conditions in community C_1 has deteriorated to sufficiently low values, the migration of people from community C_2 to community C_1 reaches almost negligible values; consequently, the average population density $\bar{\rho}_2$ again starts increasing due to the average rate of population generation $\bar{\dot{\rho}}^{(2)}$ and simultaneously the average degree of individual advancement $\bar{D}_2$ of the community starts decreasing.

Figure 3 presents the results for the variation of the average migration flux densities to the communities. During the first ten years after the establishment of the excess available wealth in community C_1, there exists substantial levels of average migration influx to community C_1 from the external surroundings C_e and the second community C_2 due to the rapid increase in the average degree of individual advancement $\bar{D}_1$ of the community. However, as the average population density $\bar{\rho}_1$ of the community increases and the average degree of individual advancement starts deteriorating, according to the conditions of the advancement potential, the migration flux density to the community rapidly decreases; and after approximately twenty years the inhabitants start migrating to the external surroundings C_e from the community C_1.

The migration out-flux from community C_2 to the external environment C_e decreases gradually as the average degree of individual advancement $\bar{D}_2$ increases. As a result of the rapid improvement of the living conditions in community C_1, during the first ten years, there exists a substantial level of migration from community C_2 to community C_1. However, as the community C_1 starts deteriorating, this migration flux density rapidly decreases and ultimately vanishes as the both communities attain approximately the same levels of advancement.

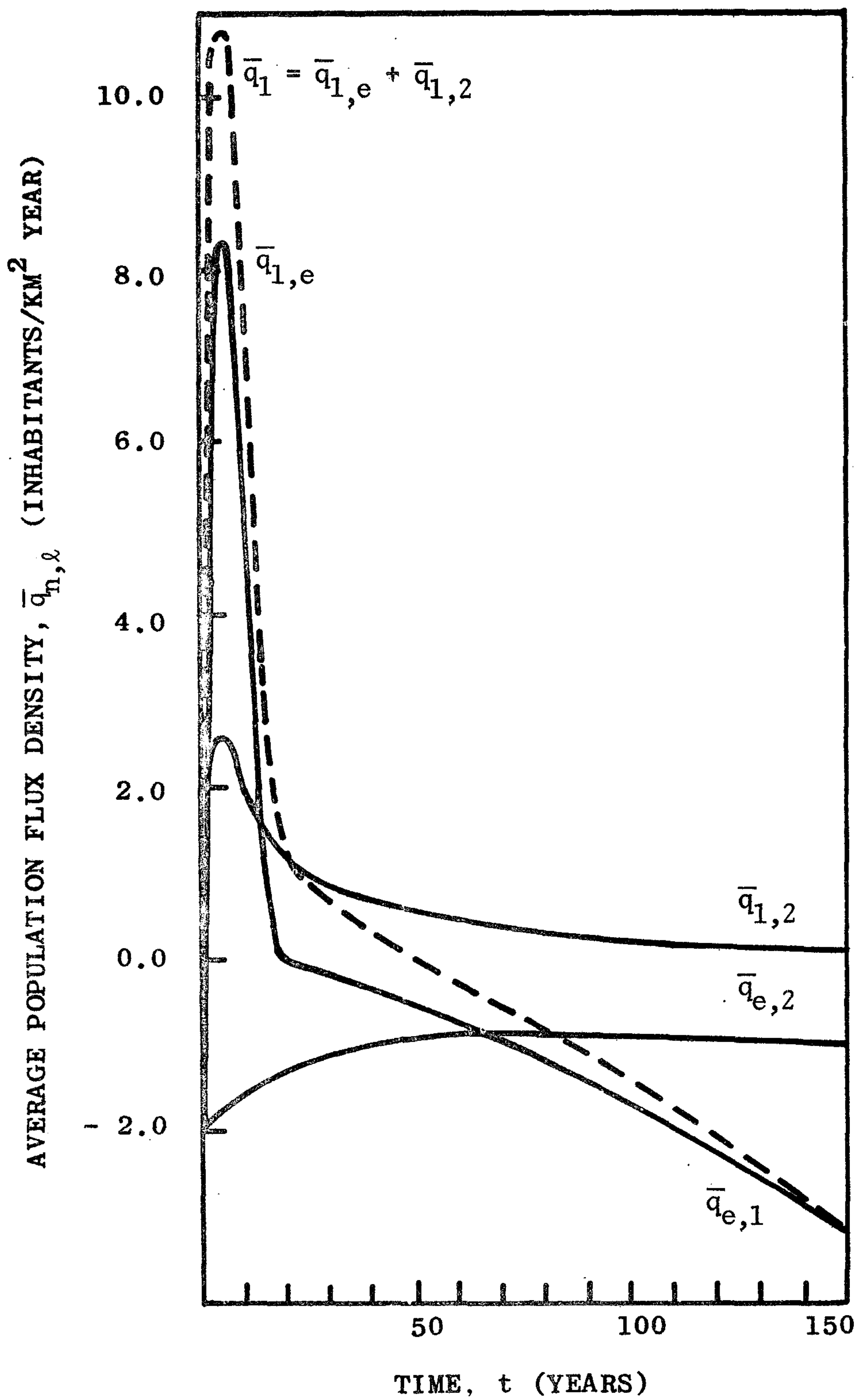

Figure 3. Variations of Average Migration Flux Density.

The net migration flux $\overline{q}_1$ to the community C_1 from the external surroundings C_e and the community C_2 increases very rapidly in the first ten years and reaches the significantly large value of approximately 11 inhabitants/km^2 year, simultaneously with the initiation of the decrease of the average degree of individual advancement, after ten years, the net migration flux $\overline{q}_1$ to the community also starts decreasing. However, for approximately fifty years the migreation flux $\overline{q}_1$ retains positive values, i.e., the inhabitants continue to move to community C_1 regardless of the rapidly deteriorating behavior of the living conditions, until the average degree of individual advancement of the community drops to the level of the other communities in the system.

GENERAL CONCLUSIONS

The general theory presents a novel mathematical formulation for the natural phenomena of human habitation between interacting communities based on a modified application of the new theories in Biophysics.

The formulation quantitatively incorporates the migration phenomena between the communities and the ultimate effects of the migrations on the overall living standards of the communities.

The average living conditions of the inhabitants are simultaneously analyzed, together with the population of the communities; hence, the proposed theory, for the first time, succeeds in incorporating the essential and continuously changing effects of the well beings of the inhabitants in the formulation of dynamic, population ekistics problems.

As consequences of the potential laws for the migration phenomena, the proposed theory concretely establishes the type of statistical data that needs to be compiled about the attitudes of the inhabitants of the communities; and furthermore, it indicates the universal applicability of this data for the analysis of the human habitation problems under different conditions of the communities.

The differential equations associated with the general theory clearly show, in mathematical, quantitative form, the potential danger of population explosion in any community to the overall interacting system. Furthermore, the differential equations which govern the living standards of the inhabitants in the communities clearly establish in quantitative form the well known quantitative hypothesis that the population explosion can be considerably more detrimental in affluent communities than in under-developed ones.

The application of the theory to a simple, hypothetical problem, about the interaction of two initially under-developed communities with each other and the large external surroundings, establish quantitative results of significant importance in relation to the development of an under-developed community by the sudden introduction of large scale financial opportunity in the particular community only.

The quantitative solutions clearly show that it is indeed futile to expect a community to attain high standards of living conditions by improving the financial opportunity only in that particular community. The unavoidable existence of the migratory phenomena between the communities always increases the population of the community to such a level that the living standard ultimately deteriorates to a considerably lower level than the originally desired one. Indeed, paradoxically, the sudden increase of the economical potential of only one community in a system of communities ultimately leads to living conditions worse than the other community's with significantly high population density and donsiderably low degree of individual advancement of the inhabitants.

Consequently, the application of the theory to a simple hypothetical problem quantitatively leads to the fundamentally important conclusion that as long as the inhabitants have the tendency to migrate to improve their well being, the living standards of a simple community can never be improved above the standards of the other communities. Indeed the improvement of the living standards in any one community can only be established by improving the living standards at an equivalent amount in all the communities of the system simultaneously.

In conclusion, the mathematical theory presents a quantitative formulation of the general dynamic population ekistics problem which can be applied to any realistic problem of any complexity for the analysis of the human habitation phenomena in any specified system of communities.

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